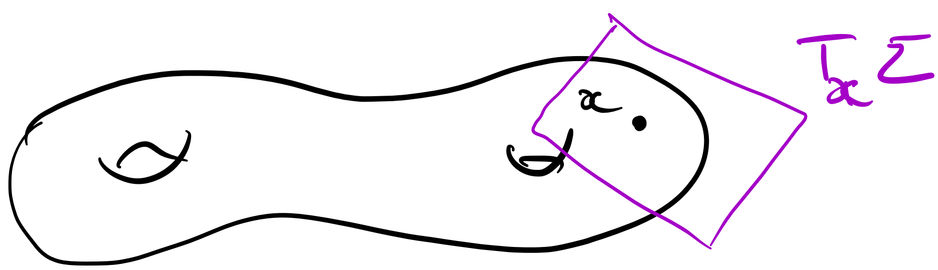
CHERN CLASSES OF COMPLEX VECTOR BUNDLES

1. Vector bundles

- The tangent bundle of a manifold.

$\Sigma \hookrightarrow \mathbb{R}^3$ a smooth surface



$T_x\Sigma$ is the plane through x tangent to the surface Σ

$$\left\{ (x, v) \in \mathbb{R}^3 \times \mathbb{R}^3 \;\middle|\; \begin{array}{l} x \in \Sigma \\ v \in T_x\Sigma \end{array} \right\}$$

is called the **TANGENT BUNDLE** of Σ.

- The tangent bundle is a particular example of **VECTOR BUNDLE**. Loosely speaking a vector bundle is a "*parametrised family of vector spaces*".

In the case of the tangent bundle of Σ, for each point x, we are given a 2-dimensional vector space $T_x M$.

We call Σ the **BASE** (i.e. the space we use to parametrise)

- for any x, $T_x\Sigma$ is the FIBER over x (the vector space parametrized by x)
- the whole tangent bundle $T\Sigma$ is the TOTAL SPACE

Finally, we've got a natural projection

$$\pi : \begin{array}{ccc} T\Sigma & \longrightarrow & \Sigma \\ (x,v) & \longmapsto & x \end{array}$$

such that $\forall x \in \Sigma$, $\pi^{-1}(\{x\}) = T_x\Sigma$.

<u>Definition</u> (FORMAL)

A VECTOR BUNDLE OVER A MANIFOLD M is a manifold E (the TOTAL SPACE) together with a map $\pi : E \longrightarrow M$ such that

1) $\forall x \in M$, $\pi^{-1}(\{x\}) := F_x$ is endowed with the structure of a vector space. F_x is THE FIBER over x.

2) $\forall x \in M$, $\exists U_x \subset M$ a neighbourhood of x in M and a diffeomorphism

$$\varphi_x : \pi^{-1}(U_x) \longrightarrow U_x \times \mathbb{R}^n$$

such that for all $y \in U_x$

$$\varphi_x^{-1} : \{y\} \times \mathbb{R}^n \longrightarrow F_y$$

is a linear isomorphism.

Examples:

- The trivial bundle $X \times \mathbb{R}^n$
- For any manifold X, TX the tangent bundle is a vector bundle.
- Consider $\mathbb{RP}^n$ (or $\mathbb{CP}^n$)

$\mathbb{RP}^n$: $\{ D$ line through the origin in $\mathbb{R}^{n+1} \}$

consider the set $\subset \mathbb{RP}^n \times \mathbb{R}^{n+1}$

$$E = \{ (D, v) \mid D \in \mathbb{RP}^n, v \in D \}$$

E is a fiber bundle over $\mathbb{RP}^n$. It is called the TAUTOLOGICAL BUNDLE

$\boxed{\text{MORPHISMS BETWEEN BUNDLES}}$

(X, E), (Y, F) two bundles
E, F total spaces, X and Y bases.

A bundle morphism is a continuous map $f : E \longrightarrow F$ such that f maps fibers of E to fibers of F.

SECTIONS OF A VECTOR BUNDLE

- A vector field on a manifold M is, by definition, the choice for any $x \in M$, of a vector in the fiber of the tangent bundle over x.

The notion of a SECTION of a vector bundle generalizes that of a vector field.

<u>Definition</u>: A SECTION of a vector bundle E over X is the datum, for any $x \in X$ of an element $s(x) \in F_x :=$ the fiber over x.

We say the section is

- continuous if $s : X \to E$ is continuous
- smooth ______________ is smooth
- whatever ______________ is whatever

2. Complex line bundles and 1st Chern class

From now on, we will concerned with the following basic question:

ARE THERE NON-TRIVIAL BUNDLES?

This is a provocative question, of course there are, otherwise we wouldn't be bothering with a theory of vector bundles. Still, since vector bundles are *locally trivial*, non-triviality of vector bundles is a global property, harder to establish.

- BUNDLES OVER CONTRACTIBLE MANIFOLDS

Proposition: Any vector bundle over a contractible space ($\mathbb{R}^n$ or the open ball in $\mathbb{R}^n$) is trivial

- BUNDLES OVER S^1

Proposition:
- There are exactly two isomorphism classes of REAL vector bundles over S^1:
  - the trivial bundle
  - the (unique) non-orientable one
- Any COMPLEX vector bundle over S^1 is trivial

Proofs of these two propositions are left as an exercise.

COMPLEX LINE BUNDLES OVER MANIFOLDS

We move to the first non-trivial task of the theory: classifying line bundles.

REAL LINE BUNDLES: we leave this case as an exercise, as the general approach is a simplified version of the complex case.

From now onwards, E is a complex line bundle over X a (real) manifold.

Important example: The tangent bundle of a COMPLEX CURVE is a complex line bundle over a 2-dim. (real) manifold.

Theorem: Let X be a manifold.
For any line bundle E over X, there exists $c_1(E) \in H^2(X,\mathbb{Z})$ such that 1) $c_1(E) = 0 \iff E$ is trivial

2) If F is another line bundle over X

$$c_1(E \otimes F) = c_1(E) + c_1(F)$$

$c_1(E)$ is called the (FIRST) CHERN CLASS of the line bundle E

We are now going to construct $c_1(E)$.
The way we are going to do this is by trying to understand what are

the OBSTRUCTIONS to trivialising a line bundle.

STRATEGY : we try our best to trivialise E, and see where and why we get stuck.

X is a smooth manifold, we consider a triangulation of it.

X_0 := 0- skeleton
X_1 := 1- skeleton
$\vdots$
X_i := i - skeleton

What we are going to do is to trivialise E inductively over the X_is

- Having trivialised E over X_i,

we can trivialise E over individual $(i+1)$- cells.

- we then check whether the trivialisation over X_i and over $(i+1)$-cells can be merged to a trivialisation over X_{i+1}.

THAT'S WHEN WE WILL SEE OBSTRUCTIONS

TRIVIALISATION OVER X_0

X_0 is a countable union of points.

$E_{|X_0}$ is obviously trivialisable

$E_{X_0} \simeq X_0 \times \mathbb{C}$

TRIVIALISATION OVER X_1

We take each 1-cell (which are intervals) and trivialise E over them (which we can as intervals are contractible)

I a 1-cell $\quad E|_I \simeq [a,b] \times \mathbb{C}$

" $[a,b]$

$\{a\}, \{b\} \in X_0$

The trivialisation of E over I in $\{a\}$ might differ from the one we have chosen, the difference being a linear map

$$\begin{array}{ccc} \{a\} \times \mathbb{C} & \longrightarrow & \{a\} \times \mathbb{C} \\ z & \longmapsto & \alpha \cdot z \end{array}$$

trivialisation over X_0 $\qquad$ trivialisation over I

If $\underset{(x,v)}{[a,b] \times \mathbb{C}}$ is a trivialisation over I, so is $(x, g(x) \cdot v)$

where $g : I \longrightarrow \mathbb{C}^*$

let β be the complex number representing the change of coordinate between trivialis° of E over I and X_0 at $\{b\}$.

It suffices to take

$$g : I \to \mathbb{C}^*, \quad g(a) = \alpha^{-1}$$
$$g(b) = \beta^{-1}$$

since $\mathbb{C}^*$ is 1-connected, this is possible.

Conclusion : $E_{|X_1}$ is always trivialasiable

TRIVIALISATION OVER X_2

. 2-cells are contractible, so we can trivialise E over any such

2 cell Δ.

- In order to turn this trivialisation into a trivialisation over X_2, we need to make the trivialisation over $\partial\Delta = S^1 \subset X_1$ agree with that over X_1.

The change of coordinates from one trivialisation to the other is of the form

$$\begin{array}{ccc} S^1 \times \mathbb{C} & \longrightarrow & S^1 \times \mathbb{C} \\ (x, v) & \longmapsto & (x, g(x)\cdot v) \end{array}$$

trivialisation over Δ | trivialisation over X_1

with $\begin{array}{ccc} S^1 & \longrightarrow & \mathbb{C}^* \\ x & \longmapsto & g(x) \end{array}$

- The trivialisation over Δ can be modified only as to continuously deform

$$x \longmapsto g(x).$$

This path is $\mathbb{C}^*$ can be made trivial
iff $[g] = 0 \in \pi_1(\mathbb{C}^*) \simeq \mathbb{Z}$

FOR EVERY 2-CELL Δ, WE HAVE DEFINED

$$c_1(\Delta) \in \mathbb{Z},$$

AN OBSTRUCTION TO EXTENDING THE TRIVIALISATION TO THE 2-SKELETON

TRIVIALISATION OVER X_i $i \geq 3$

- Using the same reasoning, to find a trivialisation on a i-cell Δ_i that on its boundary $\partial \Delta_i$ agrees ~~with~~ with a trivialisation over X_{i-1}, it is enough to continuously deform a map : $g : \partial \Delta_i = S^{i-1} \longrightarrow \mathbb{C}^*$

to the constant map 1.

Since S^{i-1}, $i \geqslant 3$ is 1-connected, this is always possible.

We have thus established the following claim.

IF WE HAVE MANAGED TO TRIVIALIZE E OVER X_2, THEN E IS TRIVIAL OVER X

We conclude with an analysis of the 2 co-chain c_1.

Fact 1: if $c_1 \equiv 0$ then $E_{|X_2}$ is trivializable

Fact 2: c_1 is a cocycle

(Hint: each 1-cell contributes positively and negatively when one computes c_1

on the boundary of a 3-cell)

Fact 3 : A change in the trivialisation of X_1 modifies c_1 by a co-boundary, and any co-boundary can be thus realised.

This way we get

- $c_1(E)$ defines an element in $H^2(X, \mathbb{Z})$
- E is trivial $\Longleftrightarrow$ $c_1(E) = 0$

Rq: This trivialisation method can also be used to show that $\forall c \in H^2(X, \mathbb{Z})$ $\exists E$ a line bundle such that

$$c_1(E) = c$$

- if E_1 and E_2 are two line bundles over X such that

$$c_1(E_1) = c_2(E_2)$$

then $E_1 \simeq E_2$

---

3. Chern classes for complex vector bundles

The first Chern class that we have just defined is just one of a sequence of cohomology classes of the base space that one can define for a complex vector bundle.

- FIRST CHERN CLASS OF A CX VECTOR BUNDLE

Let E be a complex vector bundle over a space X. By definition, its 1^{st} Chern class is

$$c_1(E) := c_1(\det E)$$

where $\det(E)$ is the determinant bundle of E

(The determinant of a bundle is the line bundle the local sections of which are volume forms on fibers of E)

Theorem / Definition

Let X be a manifold. For any complex vector bundle E over X, there are cohomology classes

$$c_i(E) \in H^{2i}(X, \mathbb{Z}) \qquad i \geq 0$$

satisfying the following axioms

1) If Y is another manifold and $f: Y \longrightarrow X$ a continuous map then $\forall E$ cx vect. bundle

$$c_1(f^*E) = f^*(c_1(E))$$

2) $\forall E, F$ vector bundles over X

$$c_n(E \oplus F) = \sum_{i+j=n} c_i(E) \cdot c_j(F)$$

3) $c_i(E) = 0$ if $i > \operatorname{rank}(E)$

4) If E is the tautological bundle over $\mathbb{C}P^1$ $c_1(E)$ generates $H^2(\mathbb{C}P^1, \mathbb{Z})$

Of course c_1 thus defined agrees with the first Chern class defined in the previous paragraph.

SKETCH OF THE CONSTRUCTION OF CHERN CLASSES

X a compact manifold.

1) A construction of bundles over X

$Gr(k,n) := \{ k\text{-planes in } \mathbb{C}^n \}$

$TGr(k,n) :=$ tautological bundle over $Gk(k,n)$

Let $f : X \longrightarrow Gr(k,n)$ a smooth map

$E := f^* \, TGr(k,n)$ defines a k-bundle over X.

Fact 1: if $f_1, f_2 : X \longrightarrow Gr(k,n)$ are homotopic, $f_1^*(TGr(k,n))$ and $f_2^*(TGr(k,n))$ are homotopic.

2) Arbitrary bundles

Fix $k \geq 0$ $\exists m = m(X) > 0$ such that every k-bundle over X is isomorphic to $f^*(T\,Gr(k,m))$ for some $f : X \longrightarrow Gr(k,m)$

3) Cohomology of $Gr(k,m)$

Recall that $H^\bullet(X,\mathbb{Z}) = \bigoplus_{i=0}^{\infty} H^i(X,\mathbb{Z})$ is a graded ring for the cup-product.

<u>FACT</u>: $H^\bullet(Gr(k,m),\mathbb{Z})$ is generated (as a ring) by k cohomology classes

$$[\alpha_i] \in H^{2i}(Gr(k,m),\mathbb{Z})$$

4) Definition of Chern classes

If E is a cx vector bundle over X such that

E := pull back of taut. bundle by a map $f : X \longrightarrow Gr(k,n)$

$$\boxed{c_i(E) := f^*([\alpha_i]) \in H^{2i}(X,\mathbb{Z})}$$

i-th Chern class of E

To make sure that these

Chern classes are well-defined one has to check all the following things:

- Any E cx vector bundle is isomorphic to the pull-back of the tautological bundle over $Gr(k,m)$ for a map $f : X \to Gr(k,m)$ for some $m \geq 0$
- Any two such maps are homotopic
- The definition doesn't depend on the value of m.

We do not prove these facts here (ref Milnor - Stasheff)